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A Model Field Theory in which Two Nambu–Goldstone Symmetries are Realised by One Massless Pseudoscalar Boson

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Abstract

We consider a model field theory consisting of two Nambu–Jona-Lasinio spin $\frac{1}{2}$ fields interacting via a coupling $f(\bar{\psi}_1\gamma^\mu\gamma^5\psi_1)(\bar{\psi}_2\gamma_\mu\gamma^5\psi_2)$ and which is therefore invariant under the two symmetries $\psi_1(x)\to e^{i\alpha_1\gamma^5}\psi_1(x)$ and $\psi_2(x)\to e^{i\alpha_2\gamma^5}\psi_2(x)$. We look for solutions in which these symmetries are spontaneously broken by giving the fermions non-zero masses. Each of the two pairs of axial-vector vertex functions in the theory satisfy two coupled integral equations, which are solved in the 'chain approximation'. We find that all four vertex functions have the same singularity structure, in particular a pole at $q^2 = 0$ corresponding to a massless pseudoscalar Nambu–Goldstone boson, and another pole corresponding to an axial-vector boson whose mass is cut-off dependent, but which for a certain range of values of f^2 is a stable particle. By considering the coupling of the strings of nucleon–antinucleon psuedoscalar 'bubbles' which generate the massless Nambu–Goldstone bosons associated with fermions 1 and 2, we show explicitly that there is only one massless Nambu–Goldstone boson in the theory.

1. *Introduction*

The concept of spontaneously broken symmetry‡ (or to give it its modern name, Nambu–Goldstone realisation of symmetry) in quantum field theory emerged in the late 1950s with the work of Heisenberg and his coworkers (Dürr *et al.*, 1959, 1961; Heisenberg, 1966) on non-linear relativistic quantum field theory, and with the BCS theory of superconductivity (Bardeen *et al.*, 1957; Bogoliubov *et al.*, 1959; Valatin, 1958; Bogoliubov, 1959; Nambu, 1960). The basic idea is that even though the Lagrangian is invariant under a certain

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‡ For a review of this topic, see Grib *et al.* (1970).

symmetry, the ground state (vacuum), and therefore the solutions of the equations of motion, may not be. The work of Goldstone (1961) and Nambu and Jona-Lasinio (1961a) on spontaneous symmetry breakdown in model field theories, in which they found that the phenomenon is accompanied by the appearance of zero mass bosons, led to the conjecture (Goldstone, 1961) that this is always the case. It was indeed subsequently proved (Goldstone *et al.*, 1962; Bludman & Klein, 1963) that spontaneous symmetry breakdown in quantum field theory always leads to the presence of zero mass bosons. In non-relativistic quantum field theory, the Goldstone theorem gave a unified view of such phenomena as phonons in a crystal, magnons (spin waves) in a ferromagnet, etc., but its application to the relativistic case was fraught with an essential difficulty. The point is that because of Lorentz invariance the massless Nambu-Goldstone bosons must have zero spin,† and although we see plenty of broken symmetries in particle physics, which we might like to try and describe by spontaneous symmetry breakdown, we do not see any massless spin-zero particles. So ever since the Goldstone theorem was proved, the theoretical efforts have been devoted to finding ways to evade it.

At the present time, there are two ways of doing this. The first is to introduce explicit symmetry-breaking terms into the Lagrangian, which give masses to the Nambu-Goldstone bosons (Nambu & Jona-Lasinio, 1961b; Glashow & Weinberg, 1968). The second mechanism is rather more subtle. It has its origins in a remark by Schwinger (1962) that the vector field which is introduced into a Lagrangian to ensure gauge invariance of the second kind need not have zero mass if the vacuum fluctuations of the matter current, $j^{\mu}(x)$, which is the source of $A^{\mu}(x)$, satisfy a certain criterion. Anderson (1963) then showed that Schwinger's criterion was equivalent to the requirement that the matter current, before the introduction of $A^{\mu}(x)$, contain a contribution from mass zero, and furthermore that a non-relativistic plasma is an example of such a theory. Relativistic examples of such theories, in which the zero-mass matter field is a Nambu-Goldstone boson, were given by Higgs (1964a, 1964b, 1966) and Englert & Brout (1964). This combining, as it were, of the massless spin-zero Nambu-Goldstone boson and the massless vector gauge field to give a massive vector field is known as the Higgs mechanism.‡ An important ingredient in these theories is the presence in the Lagrangian of a *mixing term* between the Nambu-Goldstone boson and the massless field, e.g. in the Higgs model there is a term§ $-e\lambda A_{\mu}\,\partial^{\mu}\varphi_2$ (or $-e\lambda A_{\mu}\,\partial^{\mu}\theta$ if the complex scalar field is written in polar coordinates) where $A^{\mu}(x)$ is the vector gauge field and $\varphi_2(x)$ is the massless Nambu-Goldstone boson, and in the Lagrangian density

† The situation is actually not quite as straightforward as we imply here, see Guralnik & Hagen (1968, 1969) and references therein.

‡ The Higgs mechanism, or rather its generalisation to non-abelian gauge groups (Kibble, 1967) has been made the basis of a unified theory of weak and electromagnetic interactions by Weinberg (1967, 1971) and Salam (1968). For a review of the recent extensive work on this subject see Lee (1972). Llewellyn Smith (1973), Riazuddin (1972), and Zumino (1972).

§ Throughout this paper we will use the metric, γ-matrix, and Feynman rules conventions of Bjorken & Drell (1964, 1965).

describing a plasma there is a term $\frac{1}{2}\rho_0\varphi\nabla.\mathbf{R}$ where $\varphi(x)$ is the Coulomb potential and $\mathbf{R}(x)$ is the displacement field of the electron gas (i.e. the phonon field).

Suppose we have a theory with two spontaneously broken symmetries and therefore *a priori* two massless Nambu-Goldstone bosons. We now ask the question: is it possible, by allowing mixing between these two particles, to eliminate one or both of them from the theory with the possible appearance of massive particles in their place? It is the purpose of this paper to show that there are models in which, because of mixing, it is possible to have two spontaneously broken symmetries, but with only massless Nambu-Goldstone boson.

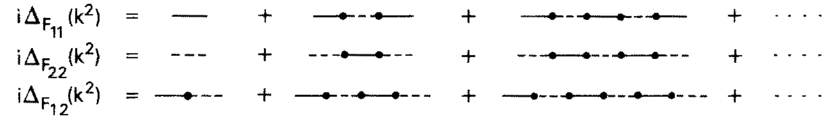

Figure 1.–The three propagators $\Delta'_{F_{11}}(k^2)$, $\Delta'_{F_{22}}(k^2)$, and $\Delta'_{F_{12}}(k^2)$. An unbroken line denotes particle 1, a dashed line denotes particle 2, and a blob denotes the vertex.

It is, of course, very easy to get rid of massless particles by mixing. Consider the Lagrangian density

$$\mathscr{L} = \tfrac{1}{2}(\partial\varphi_1)^2 + \tfrac{1}{2}(\partial\varphi_2)^2 + g\varphi_1\varphi_2 \tag{1.1}$$

where the coupling constant g is real, positive, with the dimensions of mass squared. The Feynman rules for this theory are:

(a) a factor i/k^2 for each internal 1-line of four-momentum k^μ;
(b) a factor i/k^2 for each internal 2-line of four-momentum k^μ;
(c) a factor ig for each vertex where a 1-line joins a 2-line.

It is therefore straightforward to calculate the three full propagators $\Delta'_{F_{11}}(k^2)$, $\Delta'_{F_{22}}(k^2)$, and $\Delta'_{F_{12}}(k^2)$ depicted in Fig. 1, for example:

$$i\Delta'_{F_{12}}(k^2) = ig\left(\frac{i}{k^2}\right)^2 + (ig)^3\left(\frac{i}{k^2}\right)^4 + (ig)^5\left(\frac{i}{k^2}\right)^6 + \cdots$$

i.e.

$$\Delta'_{F_{12}}(k^3) = \frac{-g}{(k^2)^2 - g^2} \tag{1.2a}$$

similarly:

$$\Delta'_{F_{11}}(k^2) = \frac{k^2}{(k^2)^2 - g^2} \tag{1.2b}$$

$$\Delta'_{F_{22}}(k^2) = \frac{k^2}{(k^2)^2 - g^2} \tag{1.2c}$$

The propagators have simple poles at $k^2 = \pm |g|$ so instead of two massless particles, we have a tardyon† of mass-squared $|g|$ and a tachyon† of mass-squared $-|g|$. In fact the particle content of the theory could have been seen more easily by making the transformation to normal modes $\Phi = (\varphi_1 + \varphi_2)/\sqrt{2}$, $\varphi = (\varphi_1 - \varphi_2)/\sqrt{2}$ in the Lagrangian (1.1). So the theory eliminates the massless particles, but only at the expense of introducing a tachyon, and since these particles have not been seen (Kreisler, 1973), this model is not of much practical importance at the present time.‡

A more interesting model is that described by the Lagrangian density

$$\mathscr{L} = \tfrac{1}{2}(\partial\varphi_1)^2 + \tfrac{1}{2}(\partial\varphi_2)^2 + g(\partial\varphi_1)\cdot(\partial\varphi_2) \tag{1.3}$$

where the coupling constant g is real positive or negative (but not equal to ± 1),§ and dimensionless. The first point to note about this Lagrangian is that it is invariant under the two symmetries|| $\varphi_1(x) \to \varphi_1(x) + \lambda_1$, $\varphi_2(x) \to \varphi_2(x) + \lambda_2$ which are spontaneously broken since they change the vacuum expectation value of the fields $\varphi_1(x)$ and $\varphi_2(x)$ respectively. The Feynman rules for the theory are:

(a) a factor i/k^2 for each internal 1-line of four-momentum k^μ;
(b) a factor i/k^2 for each internal 2-line of four-momentum k^μ;
(c) a factor igk^2 for each vertex where a 1-line of momentum k^μ turns into a 2-line of momentum k^μ.

The full propagators $\Delta'_{F_{11}}(k^2)$, $\Delta'_{F_{22}}(k^2)$, and $\Delta'_{F_{12}}(k^2)$ of Fig. 1 can be easily, calculated in this model too, and we find

$$\Delta'_{F_{12}}(k^2) = \frac{-g}{(1-g^2)k^2} \tag{1.4a}$$

$$\Delta'_{F_{11}}(k^2) = \frac{1}{(1-g^2)k^2} \tag{1.4b}$$

$$\Delta'_{F_{22}}(k^2) = \frac{1}{(1-g^2)k^2} \tag{1.4c}$$

so that *the propagators have only a single pole at* $k^2 = 0$.

† A tardyon is a particle with positive on-mass-shell mass-squared, which therefore always has a velocity less than that of light (when on-shell). A tachyon field ($\Phi(x)$) system negative on-mass-shell mass-squared, which therefore always has a velocity greater than that of light (when on-shell) (Bilaniuk & Sudarshan, 1969).

‡ Suppose, however, that we add a term $-(h/16)(\varphi_1 \pm \varphi_2)^4$ (with $h > 0$) for $g \gtrless 0$, to the Lagrangian (1.1). Then for $g > 0$ say, $\mathscr{L}$ becomes

$$\tfrac{1}{2}(\partial\Phi)^2 + \tfrac{1}{2}(\partial\varphi)^2 + \frac{g}{2}(\Phi^2 - \varphi^2) - \frac{h}{2}\Phi^4$$

and we can have spontaneous symmetry breakdown of the tachyon field ($\Phi(x)$) system to the ground state given by $\langle 0 | \Phi(x) | 0 \rangle = (g/h)^{1/2}$, from which the field $\Phi'(x) = \Phi(x) - (g/h)^{1/2}$ excites interacting particles of mass-squared $2g$ (Goldstone, 1961).

§ In the case when $g = \pm 1$, $\mathscr{L}$ just becomes equal to $\frac{1}{2}(\partial(\varphi_1 \pm \varphi_2))^2$, i.e. a free massless spin-zero field.

|| The constant field displacement, and its connection with the Goldstone theorem have been studied by Hellman & Roman (1966).

It is obvious that the pole in each of the three propagators corresponds to the same physical particle,† the propagator $\Delta'_{F_{12}}(k^2)$ differing from $\Delta'_{F_{11}}(k^2)$ and $\Delta'_{F_{22}}(k^2)$ only by the factors •––– and –––• respectively, both of which are equal to $-g$. The fields $\varphi_1(x)$ and $\varphi_2(x)$ therefore excite the same massless particle, and we have a model in which *one massless Nambu-Goldstone boson is associated with two spontaneously broken symmetries.*

The model described by equation (1.3) is, of course, rather elementary, and the remainder of this paper is devoted to a more realistic model in which there are two spontaneously broken symmetries but mixing leads to only one massless Nambu-Goldstone boson. It is a model in which two massless spin-$\frac{1}{2}$ fields, ψ_1 and ψ_2 each have an interaction of the Nambu-Jona-Lasinio type, and also interact via a mixing term of the form $f(\bar{\psi}_1\gamma^\mu\gamma^5\psi_1)(\bar{\psi}_2\gamma_\mu\gamma^5\psi_2)$, the Lagrangian density being invariant under two chiral symmetries $\psi_1(x) \to e^{i\alpha_1\gamma^5}\psi_1(x)$ and $\psi_2(x) \to e^{i\alpha_2\gamma^5}\psi_2(x)$ which we allow to be spontaneously broken by non-zero masses for the fields ψ_1 and ψ_2, which will henceforth be referred to as 'nucleons'. We show that the strings of pseudoscalar nucleon anti-nucleon bubbles, which, in the absence of the mixing term, would each give a massless pseudoscalar boson, mix via couplings of the form q^μ/q^2 and $(g^{\mu\nu} - q^\mu q^\nu/q^2)$ which satisfy $(q^\mu/q^2)(q_\mu/q^2) = 1/q^2$ and $(g^{\mu\rho} - q^\mu q^\rho/q^2) \times (g_\rho{}^\nu - q_\rho q^\nu/q^2) = (g^{\mu\nu} - q^\mu q^\nu/q^2)$ respectively, leading to a single pseudoscalar pole at $q^2 = 0$.

The layout of the paper is as follows. In Section 2, we review those aspects of the Nambu-Jona-Lasinio model which are relevant to our work. Section 2A is devoted to the self-consistent mass equation, and Section 2B to the evaluation of 'bubble' graphs and their summation to give the various mesons. In particular, we argue that the conventional (i.e. quantum electrodynamics) expression for the vector bubble used by Nambu & Jona-Lasinio (1961a, 1961b) is incorrect for this model, since the quadratically divergent part is unambiguously given by the self-consistent mass equation. In Section 2C, we write down the integral equation for the axial-vector vertex function, and solve it in the chain approximation in the two cases when the kernel is given by (a) just the pseudoscalar coupling and (b) the pseudoscalar coupling and

† It is actually instructive to study the theory described by the Lagrangian density

$$\mathscr{L} = \tfrac{1}{2}(\partial\varphi_1)^2 + \tfrac{1}{2}(\partial\varphi_2)^2 - \frac{\mu_1^2}{2}\varphi_1^2 - \frac{\mu_2^2}{2}\varphi_2^2 + g(\partial\varphi_1).(\partial\varphi_2)$$

which has propagators

$$\Delta'_{F_{12}}(k^2) = -gk^2/(1-g^2)(k^2-\mu_+^2)(k^2-\mu_-^2),$$
$$\Delta'_{F_{11}}(k^2) = (k^2-\mu_2^2)/(1-g^2)(k^2-\mu_+^2)(k^2-\mu_-^2)$$

and

$$\Delta'_{F_{22}}(k^2) = (k^2-\mu_1^2)/(1-g^2)(k^2-\mu_+^2)(k^2-\mu_-^2)$$

where

$$\mu_\pm^2 = [(\mu_1^2+\mu_2^2) \pm ((\mu_1^2-\mu_2^2)^2 + 4g^2\mu_1^2\mu_2^2)^{1/2}]/2(1-g^2)$$

first letting $\mu_2^2 \to 0$, and then $\mu_1^2 \to 0$.

the axial-vector coupling. Section 3 is devoted to the model described in the previous paragraph: in Section 3A we discuss the self-consistent mass equations for the two fermions and show that the mixing term in the Lagrangian gives no contribution to these equations in lowest order. In Section 3B, we write down the pair of coupled integral equations satisfied by the axial-vector vertex functions ${}^1\Gamma_1^{\mu 5}(p + \frac{1}{2}q, p - \frac{1}{2}q)$ and ${}^1\Gamma_2^{\mu 5}(p + \frac{1}{2}q, p - \frac{1}{2}q)$ for particles 1 and 2 respectively to couple to a particle 1-antiparticle 1 pair, and then solve them in the chain approximation. We find that the two vertex functions, and also the pair ${}^2\Gamma_1^{\mu 5}(p + \frac{1}{2}q, p - \frac{1}{2}q)$ and ${}^2\Gamma_2^{\mu 5}(p + \frac{1}{2}q, p - \frac{1}{2}q)$ for coupling to a particle 2-antiparticle 2 pair, all have the same singularity structure (see equations (3.9) and (3.10)), in particular a pole at $q^2 = 0$ corresponding to a massless pseudoscalar boson, and another pole corresponding to an axial-vector boson, whose mass is cut-off dependent, but which for small values of f^2 is an unstable particle, for medium values of f^2 is a stable particle, and for large values of f^2 is a tachyon. In Section 3C, we explicitly show, by considering the coupling of the strings of pseudoscalar nucleon–antinucleon bubbles of particles 1 and 2, that there is only one massless pseudoscalar Nambu–Goldstone boson in the theory.

In Appendix A we list some of the formulae used in evaluating bubble graphs and putting them into the form of dispersion integrals, whilst in Appendix B we show that the form of the vector bubble used by Nambu and Jona-Lasinio does not lead to sensible results for the singularities of the form-factors of the axial-vector vertex functions.

2. *The Nambu–Jona-Lasinio Model*

The Nambu-Jona-Lasinio model (Nambu & Jona-Lasinio, 1961a; Vaks & Larkin, 1961) consists of a massless spin-$\frac{1}{2}$ field with a non-linear four-fermion self-interaction described by the Lagrangian density

$$\mathscr{L} = i\bar{\psi}\partial\!\!\!/\psi + g((\bar{\psi}\psi)^2 - (\bar{\psi}\gamma^5\psi)^2) \equiv i\bar{\psi}\partial\!\!\!/\psi + \mathscr{L}_{\text{int}} \tag{2.1}$$

(where g is a real positive coupling constant with the dimensions of mass^{-2}) which is invariant under the chiral transformation $\psi(x) \to e^{i\alpha\gamma^5}\psi(x)$ as well as under the gauge transformation of the first kind $\psi(x) \to e^{i\beta}\psi(x)$ where α and β are arbitrary constants. The theory is a model of the strong interactions in which the nucleon field ψ acquires mass by spontaneous breakdown of chiral invariance, and the mesons are generated by strings of nucleon–antinucleon bubbles.

The interaction in equation (2.1) generates two types of four-fermion vertices, viz. direct and crossed (see Fig. 2), but a Fierz transformation (see Good (1955), Section VIII) on the interaction Lagrangian gives

$$\mathscr{L}_{\text{int}} = \mathscr{L}^F_{\text{int}} \equiv -\tfrac{1}{2}g((\bar{\psi}\gamma\psi)^2 - (\bar{\psi}\gamma\gamma^5\psi)^2) \tag{2.2}$$

i.e. a sum of vector and axial-vector interactions, so we can take the interaction to be $\mathscr{L}_{\text{int}} + \mathscr{L}^F_{\text{int}}$ with the understanding that we consider only the direct diagrams of Fig. 2a.

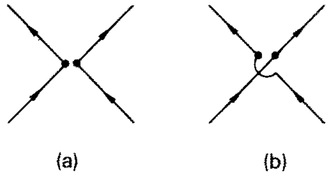

Figure 2.—The four-fermion interaction showing (a) direct and (b) crossed vertices.

The theory is, of course, unrenormalisable, and the loop integrals which occur beyond lowest order have to be given a cut-off to make them finite.

A. The Feynman Rules for the Theory with Non-Zero Mass, and the Self-Consistent Mass Equation

Nambu and Jona-Lasinio looked for solutions to the theory in which the chiral invariance is spontaneously broken by a non-zero fermion mass m i.e. for a ground state $|\Omega^{(m)}\rangle$ such that $\langle\Omega^{(m)}|\,\bar{\psi}\psi\,|\Omega^{(m)}\rangle \neq 0$. The condition for this is easily found in lowest order and is given by a generalisation of the Hartree–Fock procedure (Nambu, 1960; Bogoliubov, 1959). The Lagrangian density is written as

$$\mathscr{L} = \bar{\psi}(i\not{\partial} - m)\psi + \mathscr{L}_{\text{int}} + \mathscr{L}^{F}_{\text{int}} + m\bar{\psi}\psi \tag{2.3}$$

and the first two terms are treated as the free-field term, and the last three terms are treated as interaction terms. The Feynman rules are:

(a) a factor $2ig(1)(1)$ for each scalar vertex;
(b) a factor $-2ig(\gamma^5)(\gamma^5)$ for each pseudoscalar vertex;
(c) a factor $-ig(\gamma^\mu)(\gamma_\mu)$ for each vextor vertex;
(d) a factor $ig(\gamma^\mu\gamma^5)(\gamma_\mu\gamma^5)$ for each axial-vector vertex;
(e) a factor im for each mass term;
(f) a factor $iS_F(p, m) = i/(\not{p} - m + i\epsilon)$ for each internal fermion line;
(g) a factor -1 for each closed fermion loop.

On requiring that the lowest order corrections to the propagator should cancel (see Fig. 3) and noting that the pseudoscalar, vector, and axial-vector terms give zero contribution, we find

$$m = \frac{8igm}{(2\pi)^4}\int \frac{d^4p}{p^2 - m^2 + i\epsilon} \tag{2.4}$$

The solution $m = 0$ corresponds to the 'ordinary' solution of equation (2.1). The other solution is

$$1 = \frac{8ig}{(2\pi)^4}\int \frac{d^4p}{p^2 - m^2 + i\epsilon} \tag{2.5}$$

and is the condition for the occurrence of a spontaneously broken chiral symmetry solution of equation (2.1) with mass m and is called *the self-*

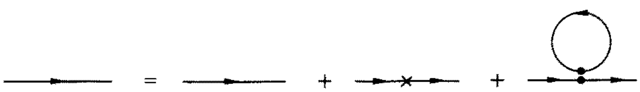

Figure 3.—The self-consistent mass equation. The lines denote a fermion of mass m.

consistent mass equation. The integral in equation (2.5) is quadratically divergent, but on cutting off with a large mass Λ^2 it can be explicitly evaluated (Nambu & Jona-Lasinio, 1961a) to give the condition on g for the mass m solution to exist, viz. $g\Lambda^2 > 2\pi^2$, but we will not reproduce this here. Nambu and Jona-Lasinio also showed that the spontaneous symmetry breakdown solution is, in fact, the stable one since the vacuum $|\Omega^{(m)}\rangle$ has a lower energy than the 'ordinary' vacuum.

B. Mesons from Strings of Bubbles

To discover what mesons are predicted by the theory, Nambu and Jona-Lasinio looked for bound states in nucleon-antinucleon scattering, or, equivalently, for exchanged particles in nucleon-nucleon scattering. The analogue of the 'ladder approximation' in nucleon-antinucleon scattering in this model is the 'chain approximation', i.e. the iteration of a nucleon-antinucleon closed loop (see Fig. 4) in which the lines have mass m. They

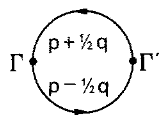

Figure 4.—The nucleon-antinucleon loop $J_{\Gamma\Gamma'}(q)$.

ascribed significance only to poles below $q^2 = 4m^2$. We define the function $\tilde{J}_{\Gamma\Gamma'}(q)$ where Γ and Γ' are Dirac matrices by† (see Fig. 4):

$$\tilde{J}_{\Gamma\Gamma'}(q) = \int \frac{\mathrm{Tr}((\not p - \tfrac{1}{2}\not q + m)\Gamma(\not p + \tfrac{1}{2}\not q + m)\Gamma')\,d^4p}{((p-\tfrac{1}{2}q)^2 - m^2)((p+\tfrac{1}{2}q)^2 - m^2)(2\pi)^4} \tag{2.6}$$

In the case where Γ and Γ' are the unit matrix we obtain, on using equation (A.4) of Appendix A:

$$\tilde{J}_{SS}(q) = \frac{4}{(2\pi)^4}\int \frac{d^4p}{p^2 - m^2} - 2(q^2 - 4m^2)I(q^2) \tag{2.7}$$

which, on using the self-consistent mass equation (2.5), gives

$$J_{SS}(q) = 2ig\tilde{J}_{SS}(q) = 1 - 4ig(q^2 - 4m^2)I(q^2) \tag{2.8}$$

So the sum of scalar bubbles‡

$$2ig1\,\frac{1}{1 - J_{SS}(q)}\,1 = 1\,\frac{1}{2(q^2 - 4m^2)I(q^2)}\,1 \tag{2.9}$$

† Henceforth, an $i\epsilon$ in each propagator denominator will be understood.

‡ The γ-matrix (in this case the unit matrix) on each side of the sums of bubbles is understood to be sandwiched between appropriate Dirac spinors.

has a pole at $q^2 = 4m^2$ corresponding to a scalar meson. When $\Gamma = \Gamma' = \gamma^5$ we obtain, on using equation (A.4):

$$\tilde{J}_{PP}(q) = 2q^2 I(q^2) - \frac{4}{(2\pi)^4}\int \frac{d^4p}{p^2 - m^2} \tag{2.10}$$

and on using the self-consistent mass equation (2.5) we obtain

$$J_{PP}(q) = -2ig\tilde{J}_{PP}(q) = 1 - 4igq^2 I(q^2) \tag{2.11}$$

So the sum of pseudoscalar bubbles

$$-2ig\gamma^5 \frac{1}{1 - J_{PP}(q)}\gamma^5 = -\gamma^5 \frac{1}{2q^2 I(q^2)}\gamma^5 \tag{2.12}$$

has a pole at $q^2 = 0$, which is just the massless pseudoscalar meson predicted by the Goldstone theorem.†

The case of the vector bubble is rather more complicated than the previous two examples. With $\Gamma = \gamma^\mu$, $\Gamma' = \gamma^\nu$, we find

$$\tilde{J}^{\mu\nu}_{VV}(q) = \int \frac{(-4p^2 + q^2 + 4m^2)g^{\mu\nu} + 8p^\mu p^\nu - 2q^\mu q^\nu}{((p - \frac{1}{2}q)^2 - m^2)((p + \frac{1}{2}q)^2 - m^2)} \frac{d^4p}{(2\pi)^4} \tag{2.13}$$

which on using equation (A.7) gives:

$$\tilde{J}^{\mu\nu}_{VV}(q) = \int \frac{(-4p^2 + q^2 + 4m^2)g^{\mu\nu} + 2p^2 g^{\mu\nu} - 2q^\mu q^\nu}{((p - \frac{1}{2}q)^2 - m^2)((p + \frac{1}{2}q)^2 - m^2)} \frac{d^4p}{(2\pi)^4}$$

$$+ A(q^2, m^2)(q^\mu q^\nu - \tfrac{1}{4}q^2 g^{\mu\nu}) \tag{2.14}$$

where $A(q^2, m^2)$ is some function of q^2 and m^2. Then current conservation, $q_\mu \tilde{J}^{\mu\nu}_{VV}(q) = 0$ gives:

$$\frac{3q^2}{4}A(q^2, m^2) = -\int \frac{(-2p^2 - q^2 + 4m^2)}{((p - \frac{1}{2}q)^2 - m^2)((p + \frac{1}{2}q)^2 - m^2)} \frac{d^4p}{(2\pi)^4} \tag{2.15}$$

† Umezawa (1965a, 1965b) has established an interesting connection between the invariance of the original Lagrangian (2.1) under chiral transformations of the fermion field and the invariance of the final effective Lagrangian under constant field displacements of the massless pseudoscalar field. This concept of dynamical rearrangement of symmetry has been applied to the Goldstone model by Nakagawa *et al.* (1966) and to the BCS model by Leplae & Umezawa (1966). A review of this and more recent work has been given by Umezawa (1973).

so, on substituting equation (2.15) into equation (2.14) and using equation (A.4) we find

$$\tilde{J}^{\mu\nu}_{VV}(q) = \frac{4}{3}\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right)\left(-\frac{2}{(2\pi)^4}\int \frac{d^4p}{p^2 - m^2} + (q^2 + 2m^2)I(q^2)\right) \tag{2.16}$$

and, on using the self-consistent mass equation (2.5), we obtain

$$\begin{aligned} J^{\mu\nu}_{VV}(q) &= -ig\tilde{J}^{\mu\nu}_{VV}(q) = \left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right)\left(\frac{1}{3} - \frac{4ig}{3}(q^2 + 2m^2)I(q^2)\right) \\ &\equiv \left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right) J_V(q^2) \end{aligned} \tag{2.17}$$

The sum of vector bubbles becomes, using the fact that $(g^{\mu\nu} - q^\mu q^\nu/q^2)$ is idempotent,

$$\begin{aligned} &-ig\gamma^\mu\left(g_{\mu\nu} + \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right)\sum_{n=1}^{\infty}(J_V(q^2))^n\right)\gamma^\nu \\ &\qquad = -ig\left(\gamma^\mu \frac{1}{1 - J_V(q^2)}\gamma_\mu - \not{q}\frac{J_V(q^2)}{(1 - J_V(q^2))q^2}\not{q}\right) \end{aligned} \tag{2.18}$$

where the second term is zero because of current conservation. It is seen that, unlike the scalar and pseudoscalar cases, the pole in the scattering amplitude does not occur straightforwardly. To exhibit it, we use the dispersive forms for $I(q^2)$ and the self-consistent mass equation, given by equations (A.10) and (A.15) respectively, and find

$$J_V(q^2) = \frac{g}{12\pi^2}\int_{4m^2}^{\Lambda^2} (1 - 4m^2/\kappa^2)^{1/2}\left(1 + \frac{(q^2 + 2m^2)}{\kappa^2 - q^2}\right) d\kappa^2 \tag{2.19a}$$

$$1 - J_V(q^2) = \frac{g}{12\pi^2}\int_{4m^2}^{\Lambda^2} (1 - 4m^2/\kappa^2)^{1/2}\left(2 - \frac{(q^2 + 2m^2)}{\kappa^2 - q^2}\right) d\kappa^2 \tag{2.19b}$$

The integrand of equation (2.19b) has a zero at $q^2 = 2(\kappa^2 - m^2)/3$, so $1/(1 - J_V(q^2))$ has a pole somewhere in the range $2m^2 \leqslant q^2 \leqslant 2(\Lambda^2 - m^2)/3$ which for sufficiently small Λ^2 will lie below $q^2 = 4m^2$, and will correspond to a stable vector meson. Note that the result which we obtain here is different from that of Nambu & Jona-Lasinio (1961a, equation (4.15)) who find that the mass squared of the vector meson is greater than $8m^2/3$. This is because they apply the additional condition, familiar from quantum electrodynamics, that $J_V(0) = 0$. This seems to us to be incorrect here for the following reasons. Firstly, in quantum electrodynamics, the condition is applied in order to guarantee that the photon has zero renormalised mass in perturbation theory; this requirement does not occur in this theory. Secondly, the quadratically divergent part of $J_V(q^2)$ is uniquely given by self-consistent mass equations (2.5) and (A.15). In fact $J_V(0)$ is just

$$J_V(0) = \frac{g}{12\pi^2}\left(\int_{4m^2}^{\Lambda^2}(1-4m^2/\kappa^2)^{1/2}\,d\kappa^2 + 2m^2\int_{4m^2}^{\Lambda^2}\frac{(1-4m^2/\kappa^2)^{1/2}}{\kappa^2}d\kappa^2\right) \tag{2.20}$$

which is a positive definite quantity. So in the Nambu-Jona-Lasinio model with the particular cut-off procedure used, it appears to be incorrect to require that $J_V(0) = 0$. In fact, when we come to consider the axial-vector vertex functions of the model to be discussed in Section 3B, we shall find that, when we use the Nambu-Jona-Lasinio form for $J_V(q^2)$, we do not obtain sensible physical results.

We next evaluate the axial-vector bubble given by equation (2.6) with $\Gamma = \gamma^\mu\gamma^5$, $\Gamma' = \gamma^\nu\gamma^5$, and find

$$\tilde{J}^{\mu\nu}_{AA}(q) = \tilde{J}^{\mu\nu}_{VV}(q) - 8m^2 g^{\mu\nu} I(q^2) \tag{2.21}$$

and, on using equation (2.16) for $\tilde{J}^{\mu\nu}_{VV}(q)$, we obtain

$$\tilde{J}^{\mu\nu}_{AA}(q) = \frac{4}{3}\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right)\left(-\frac{2}{(2\pi)^4}\int\frac{d^4p}{p^2-m^2} + (q^2-4m^2)I(q^2)\right)$$
$$-\frac{q^\mu q^\nu}{q^2}\,8m^2 I(q^2) \tag{2.22}$$

and, on using the self-consistent mass equation (2.5), we obtain

$$J^{\mu\nu}_{AA}(q) = ig\tilde{J}^{\mu\nu}_{AA}(q) = \left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right)\left(-\frac{1}{3} + \frac{4ig}{3}(q^2-4m^2)I(q^2)\right)$$
$$-\frac{q^\mu q^\nu}{q^2}\,8igm^2 I(q^2)$$
$$\equiv \left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right)J_A(q^2) - \frac{q^\mu q^\nu}{q^2}J'_A(q^2) \tag{2.23}$$

so that $J_A(q^2) = -(J_V(q^2) + J'_A(q^2))$. On summing the axial vector bubbles, we find

$$ig\gamma^\mu\gamma^5\left(g_{\mu\nu} + \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right)\sum_{n=1}^{\infty}(J_A(q^2))^n + \frac{q_\mu q_\nu}{q^2}\sum_{n=1}^{\infty}(-J'_A(q^2))^n\right)\gamma^\nu\gamma^5$$

$$= ig\left(\gamma^\mu\gamma^5\frac{1}{1-J_A(q^2)}\gamma_\mu\gamma^5 + \not{q}\gamma^5\frac{J_V(q^2)/q^2}{(1-J_A(q^2))(1+J'_A(q^2))}\not{q}\gamma^5\right) \quad (2.24)$$

We note that the second term has a pole at $q^2 = 0$ corresponding to the exchange of the pseudoscalar meson. To discover whether or not $1/(1 - J_A(q^2))$ has a pole, we write the denominator as a dispersion integral, using equations (2.23), (A.10), and (A.15):

$$J_A(q^2) = -\frac{g}{12\pi^2}\int_{4m^2}^{\Lambda^2}\frac{(1-4m^2/\kappa^2)^{1/2}(\kappa^2-4m^2)\,d\kappa^2}{\kappa^2-q^2} \quad (2.25a)$$

$$1 - J_A(q^2) = \frac{g}{12\pi^2}\int_{4m^2}^{\Lambda^2}(1-4m^2/\kappa^2)^{1/2}\left(3 + \frac{(\kappa^2-4m^2)}{\kappa^2-q^2}\right)d\kappa^2 \quad (2.25b)$$

The integrand of equation (2.25b) has a zero at $q^2 = 4(\kappa^2 - m^2)/3$, so $1/(1 - J_A(q^2))$ has a pole somewhere in the range $4m^2 < q^2 \leqslant 4(\Lambda^2 - m^2)/3$, which does not correspond to a stable particle. The expression $(1 + J'_A(q^2))$, when written in dispersive form using equations (2.23), (A.10), and (A.15), gives:

$$1 + J'_A(q^2) = \frac{g}{4\pi^2}\int_{4m^2}^{\Lambda^2}(1-4m^2/\kappa^2)^{1/2}\left(1 - \frac{2m^2}{\kappa^2-q^2}\right)d\kappa^2 \quad (2.26)$$

and the integrand has a zero at $q^2 = \kappa^2 - 2m^2$, and so $1/(1 + J'_A(q^2))$ has a pole somewhere in the range $2m^2 \leqslant q^2 \leqslant \Lambda^2 - 2m^2$, which we might think shows that there is a massive pseudoscalar† meson in the theory. However, we have so far not mentioned the fact that, because $\tilde{J}^\mu_{PA}(q)$ is non-zero, there will be mixing between the strings of pseudoscalar and axial-vector bubbles. In Section 2C, we shall take this into consideration when studying the axial-vector vertex function, and shall find that the pole $1/(1 + J'_A(q^2))$ is not, in fact, present. The explicit expression for $\tilde{J}^\mu_{PA}(q)$ is:

$$\tilde{J}^\mu_{PA}(q) = \int\frac{\mathrm{Tr}\,((\not{p}-\frac{1}{2}\not{q}+m)\gamma^5(\not{p}+\frac{1}{2}\not{q}+m)\gamma^\mu\gamma^5)}{((p-\frac{1}{2}q)^2-m^2)((p+\frac{1}{2}q)^2-m^2)}\frac{d^4p}{(2\pi)^4} = 4mq^\mu I(q^2) \quad (2.27)$$

† Pseudoscalar since the wavefunction is $\not{q}\gamma^5$.

Note also that

$$\tilde{J}^{\mu}_{AP}(q) = \int \frac{\mathrm{Tr}\,((\not p - \frac{1}{2}\not q + m)\gamma^{\mu}\gamma^{5}(\not p + \frac{1}{2}\not q + m)\gamma^{5})}{((p - \frac{1}{2}q)^{2} - m^{2})((p + \frac{1}{2}q)^{2} - m^{2})} \frac{d^{4}p}{(2\pi)^{4}}$$
$$= -4mq^{\mu}I(q^{2}) \tag{2.28}$$

All the other bubbles for which $\Gamma \neq \Gamma'$ are zero.

C. The Axial Vector Vertex Function

The first point to note about the axial vector vertex function $\Gamma^{\mu 5}(p + \frac{1}{2}q, p - \frac{1}{2}q)$ is that because of axial current conservation and Lorentz invariance it must satisfy (Nambu & Jona-Lasinio, 1961a):

$$\bar{u}(p + \tfrac{1}{2}q)\Gamma^{\mu 5}(p + \tfrac{1}{2}q, p - \tfrac{1}{2}q)u(p - \tfrac{1}{2}q)$$
$$= f(q^{2})\bar{u}(p + \tfrac{1}{2}q)\left(\gamma^{\mu} - \frac{2mq^{\mu}}{q^{2}}\right)\gamma^{5}u(p - \tfrac{1}{2}q) \tag{2.29}$$

where the pole at $q^2 = 0$ is due to the massless pseudoscalar Nambu-Goldstone boson, and $f(q^2)$ is a form factor.

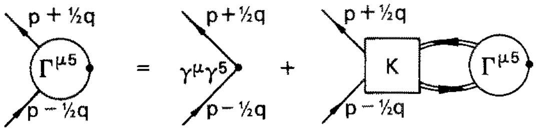

Figure 5.—The integral equation for $\Gamma^{\mu 5}(p + \frac{1}{2}q, p - \frac{1}{2}q)$.

The integral equation for $\Gamma^{\mu 5}(p + \frac{1}{2}q, p - \frac{1}{2}q)$ is shown in Fig. 5, where $K(p, p', q)$ is the nucleon-antinucleon scattering kernel. The analogue of the 'ladder approximation' in this case is to replace the full propagators by free field propagator of mass m, and to approximate K by the four-fermion interaction. On replacing K by just the pseudoscalar interaction we obtain for the integral equation:

$$\Gamma^{\mu 5}(p + \tfrac{1}{2}q, p - \tfrac{1}{2}q) = \gamma^{\mu}\gamma^{5} + 2ig\gamma^{5} \int \mathrm{Tr}\,(\gamma^{5}iS_{F}(p' + \tfrac{1}{2}q, m) \times$$
$$\Gamma^{\mu 5}(p' + \tfrac{1}{2}q, p' - \tfrac{1}{2}q)iS_{F}(p' - \tfrac{1}{2}q, m))\frac{d^{4}p'}{(2\pi)^{4}} \tag{2.30}$$

The most general ansatz for $\Gamma^{\mu 5}(p + \frac{1}{2}q, p - \frac{1}{2}q)$ satisfying equation (2.29) is

$$\Gamma^{\mu 5} = \left(\gamma^{\mu} - \frac{2mq^{\mu}}{q^{2}}\right)\gamma^{5}F(q^{2}) + \left(\gamma^{\mu} - \frac{\not q q^{\mu}}{q^{2}}\right)\gamma^{5}G(q^{2}) \tag{2.31}$$

and on substituting into equation (2.30), using the expressions for $\tilde{J}_{PP}(q)$ and $\tilde{J}^{\mu}_{PA}(q)$ viz. equations (2.10) and (2.27), and the self-consistent mass equation (2.5), we obtain

$$\left(\gamma^\mu - \frac{2mq^\mu}{q^2}\right)\gamma^5 F(q^2) + \left(\gamma^\mu - \frac{\not{q}q^\mu}{q^2}\right)\gamma^5 G(q^2) = \gamma^\mu\gamma^5 - \frac{2mq^\mu}{q^2}\gamma^5 F(q^2) \tag{2.32}$$

which tells us that $G(q^2) = 0$ and $F(q^2) = 1$. So $\Gamma^{\mu 5}(p + \frac{1}{2}q, p - \frac{1}{2}q)$ is just

$$\Gamma^{\mu 5}(p + \tfrac{1}{2}q, p - \tfrac{1}{2}q) = \left(\gamma^\mu - \frac{2mq^\mu}{q^2}\right)\gamma^5 \tag{2.33}$$

so that in this approximation the form factor $f(q^2)$ in equation (2.29) is unity. In fact the equation (2.30) on iteration simply generates all possible strings of pseudoscalar bubbles, giving the pole at $q^2 = 0$ as the only singularity.

When we use both the pseudoscalar and axial-vector interactions in K the integral equation becomes:

$$\begin{aligned}
&\Gamma^{\mu 5}(p + \tfrac{1}{2}q, p - \tfrac{1}{2}q) \\
&\quad = \gamma^\mu\gamma^5 + 2ig\gamma^5 \int \mathrm{Tr}(\gamma^5 iS_F(p' + \tfrac{1}{2}q, m)\Gamma^{\mu 5}(p' + \tfrac{1}{2}q, p' - \tfrac{1}{2}q) \\
&\quad \times iS_F(p' - \tfrac{1}{2}q, m))\frac{d^4p'}{(2\pi)^4} - ig\gamma_\nu\gamma^5 \int \mathrm{Tr}(\gamma^\nu\gamma^5 iS_F(p' + \tfrac{1}{2}q, m) \times \\
&\quad \Gamma^{\mu 5}(p' + \tfrac{1}{2}q, p' - \tfrac{1}{2}q)\, iS_F(p' - \tfrac{1}{2}q, m))\frac{d^4p'}{(2\pi)^4}
\end{aligned} \tag{2.34}$$

and on using equations (2.5), (2.10), (2.23), and (2.27), we find

$$\begin{aligned}
&\left(\gamma^\mu - \frac{2mq^\mu}{q^2}\right)\gamma^5 F(q^2) + \left(\gamma^\mu - \frac{\not{q}q^\mu}{q^2}\right)\gamma^5 G(q^2) \\
&\quad = \gamma^\mu\gamma^5 - \frac{2mq^\mu}{q^2}\gamma^5 F(q^2) + \left(\gamma^\mu - \frac{\not{q}q^\mu}{q^2}\right)\gamma^5 J_A(q^2)(F(q^2) + G(q^2))
\end{aligned} \tag{2.35}$$

giving us:

$$F(q^2) = 1 \qquad G(q^2) = \frac{J_A(q^2)}{1 - J_A(q^2)} \tag{2.36}$$

so between nucleon spinors, $\Gamma^{\mu 5}(p + \frac{1}{2}q, p - \frac{1}{2}q)$ becomes

$$\begin{aligned}
&\bar{u}(p + \tfrac{1}{2}q)\Gamma^{\mu 5}(p + \tfrac{1}{2}q, p - \tfrac{1}{2}q)\bar{u}(p - \tfrac{1}{2}q) \\
&\quad = \bar{u}(p + \tfrac{1}{2}q)\left(\gamma^\mu - \frac{2mq^\mu}{q^2}\right)\gamma^5 u(p - \tfrac{1}{2}q)\left(\frac{1}{1 - J_A(q^2)}\right)
\end{aligned} \tag{2.37}$$

and we have shown that there is no pole from $1/(1 + J'_A(q^2))$ as we mentioned at the end of Section 2B.

3. *The Model*

We now consider a field theory consisting of two Nambu-Jona-Lasinio fields† ψ_1 and ψ_2 interacting via an axial-vector coupling:

$$
\begin{aligned}
\mathscr{L} = {} & i\bar{\psi}_1 \not{\partial} \psi_1 + g_1((\bar{\psi}_1\psi_1)^2 - (\bar{\psi}_1\gamma^5\psi_1)^2) \\
& + i\bar{\psi}_2 \not{\partial} \psi_2 + g_2((\bar{\psi}_2\psi_2)^2 - (\bar{\psi}_2\gamma^5\psi_2)^2) \\
& + f(\bar{\psi}_1\gamma^\mu\gamma^5\psi_1)(\bar{\psi}_2\gamma_\mu\gamma^5\psi_2)
\end{aligned} \tag{3.1}
$$

where g_1 and g_2 are real positive coupling constants with the dimensions of mass^{-2}, and f is a real coupling constant with dimension mass^{-2}. The Lagrangian density (3.1) is invariant under the two chiral transformations

$$
\begin{aligned}
\psi_1(x) &\to e^{i\alpha_1\gamma^5}\,\psi_1(x) \\
\psi_2(x) &\to e^{i\alpha_2\gamma^5}\,\psi_2(x)
\end{aligned} \tag{3.2}
$$

where α_1 and α_2 are arbitrary constants, and is also invariant under the gauge transformations of the first kind $\psi_1(x) \to e^{i\beta_1}\psi_1(x)$ and $\psi_2(x) \to e^{i\beta_2}\psi_2(x)$ where β_1 and β_2 are arbitrary constants.

A Fierz transformation‡ on the scalar and pseudoscalar self-interactions of each particle allows them to be written as a sum of vector and axial vector terms as in equation (2.2). As in Section 2, we shall use the sum of these interactions, with the understanding that we consider only the direct diagrams of Fig. 2a.

A. The Feynman Rules for the Theory with Non-Zero Fermion Masses, and the Self-Consistent Mass Equations

We now look for solutions to the theory, described by the Lagrangian density (3.1) for which the chiral symmetries (3.2) are spontaneously broken by non-zero fermion masses m_1 and m_2. The Feynman rules for such a theory

† Suzuki (1963) has considered a model in which two Nambu-Jona-Lasinio fields interact via a mixing term. His model, in which $g_1 = g_2$ and the mixing term is $((\bar{\psi}_1\psi_1)(\bar{\psi}_2\psi_2) - (\bar{\psi}_1\gamma^5\psi_1)(\bar{\psi}_2\gamma^5\psi_2))$ is invariant under the single chiral transformation $\psi_1(x) \to e^{i\alpha\gamma^5}\psi_1(x)$, $\psi_2(x) \to e^{i\alpha\gamma^5}\psi_2(x)$ and the permutation $\psi_1 \leftrightarrow \psi_2$, not under the chiral transformations (3.2).

‡ A Fierz transformation on the mixing term yields

$$
\begin{aligned}
&(\bar{\psi}_1\gamma^\mu\gamma^5\psi_1)(\bar{\psi}_2\gamma_\mu\gamma^5\psi_2) \\
&\quad = (\bar{\psi}_1\psi_2)(\bar{\psi}_2\psi_1) - (\bar{\psi}_1\gamma^5\psi_2)(\bar{\psi}_2\gamma^5\psi_1) + \tfrac{1}{2}(\bar{\psi}_1\gamma^\mu\psi_2)(\bar{\psi}_2\gamma_\mu\psi_1) \\
&\quad\quad + \tfrac{1}{2}(\bar{\psi}_1\gamma^\mu\gamma^5\psi_2)(\bar{\psi}_2\gamma_\mu\gamma^5\psi_1)
\end{aligned}
$$

so that there are likely to be bound state of $\bar{1}2$ and $\bar{2}1$ pairs in the theory too. However these effects are crucially dependent upon the sign of f, as well as its magnitude, and can be treated separately from the mixing which depends only on f^2, and is the concern of this paper.

are the same as those given in Section 2A, except that g and m are replaced by g_a and m_a (where $a = 1, 2$), and that there is an extra rule: viz. a factor $if(\gamma^\mu\gamma^5)(\gamma_\mu\gamma^5)$ at each vertex where particles 1 and 2 interact via the coupling $f(\bar{\psi}_1\gamma^\mu\gamma^5\psi_1)(\bar{\psi}_2\gamma_\mu\gamma^5\psi_2)$.

The self-consistent mass equation for, say, particle 1, in lowest order (one-loop-approximation) is again given by Fig. 3, but there is now an extra term from a particle 2 loop, $-if\gamma_\mu\gamma^5 \int d^4p \operatorname{Tr}(\gamma^\mu\gamma^5 iS_F(p, m_2))/(2\pi)^4$ which is, however, zero. So the condition for a solution ψ_1 which spontaneously breaks the chiral symmetries (3.2) with a mass m_1, is just given by equation (2.5):

$$1 = \frac{8ig_1}{(2\pi)^4} \int \frac{d^4p}{p^2 - m_1^2} \tag{3.3a}$$

Similarly the condition for a solution ψ_2 which spontaneously breaks the chiral symmetries (3.2) with a mass m_2 is just

$$1 = \frac{8ig_2}{(2\pi)^4} \int \frac{d^4p}{p^2 - m_2^2} \tag{3.3b}$$

The dispersive form of the self-consistent mass equation (A.15) and the dispersion integral (A.10) associated with particles 1 and 2, are cut off at masses Λ_1^2 and Λ_2^2 respectively.

B. The Axial-Vector Vertex Functions

Since the mixing term in equation (3.1) is axial-vector, and recalling that we mentioned in Section 2B that there is no coupling between axial-vector and scalar or vector coupling, i.e. $\tilde{J}^\mu_{SA}(q) = 0 = \tilde{J}^{\mu\nu}_{VA}(q)$, then we can immediately say that each fermion field ψ_a has associated with it a scalar boson of mass^2 $4m_a^2$ and a vector boson with mass μ_{V_a} in the range $2m_a^2 \leqslant \mu^2_{V_a} \leqslant 2(\Lambda_g^2 - m_a^2)/3$.

However, since $\tilde{J}^\mu_{PA}(q) \neq 0$ it is clear that the pseudoscalar and axial-vector coupling of the two particles will get jumbled, for example a stream of pseudoscalar bubbles of particle 1 can couple to a stream of pseudoscalar bubbles of particle 2 via the term $f(\bar{\psi}_1\gamma^\mu\gamma^5\psi_1)(\bar{\psi}_2\gamma_\mu\gamma^5\psi_2)$. In fact, each particle a will now have *two* axial-vector vertex functions ${}^b\Gamma^{\mu5}_a$ corresponding to coupling to a $b\bar{b}$ pair where $b = 1, 2$ (see Fig. 6). These vertex functions will satisfy two pairs of coupled integral equations, the pair for ${}^1\Gamma^{\mu5}_1(p + \frac{1}{2}q, p - \frac{1}{2}q)$ and

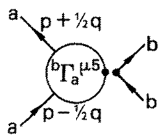

Figure 6.—The vertex function ${}^b\Gamma^{\mu5}_a(p + \frac{1}{2}q, p - \frac{1}{2}q)$, where a and b are equal to 1 or 2.

${}^1\Gamma_2^{\mu5}(p+\frac{1}{2}q, p-\frac{1}{2}q)$ being shown in Fig. 7. Clearly, our task now is to solve these equations in the 'chain approximation' and by looking at the singularities of the vertex functions, ascertain what pseudoscalar and axial-vector mesons appear in the theory.

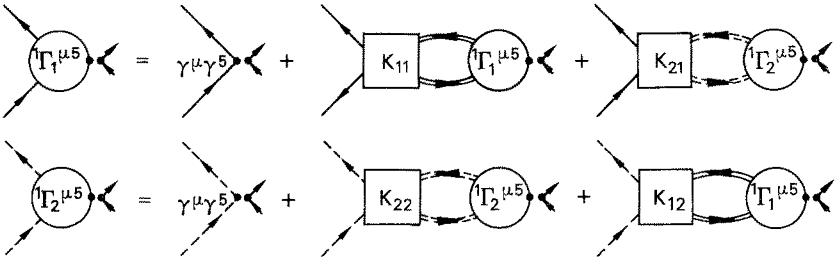

Figure 7.–The coupled integral equations for ${}^1\Gamma_1^{\mu5}(p+\frac{1}{2}q, p-\frac{1}{2}q)$ and ${}^1\Gamma_2^{\mu5}(p+\frac{1}{2}q, p-\frac{1}{2}q)$. An unbroken line denotes particle 1 and a dashed line denotes particle 2.

Replacing the full propagators $S_F'(k, m_a)$ by free-field propagators $S_F(k, m_a)$, the kernels K_{11} and K_{22} by sums of pseudoscalar and axial-vector four-fermion interactions as in Section 2C, and the kernels K_{12} and K_{21} by $if(\gamma^\mu\gamma^5)(\gamma_\mu\gamma^5)$ we obtain for the equations shown in Fig. 7:

$$
\begin{aligned}
&g_1{}^1\Gamma_1^{\mu5}(p+\tfrac{1}{2}q, p-\tfrac{1}{2}q)\\
&\quad= g_1\gamma^\mu\gamma^5 + 2ig_1{}^2\gamma^5\int \mathrm{Tr}\,(\gamma^5 iS_F(p'+\tfrac{1}{2}q, m_1)^1\Gamma_1^{\mu5}(p'+\tfrac{1}{2}q, p'-\tfrac{1}{2}q)\\
&\quad\times iS_F(p'-\tfrac{1}{2}q, m_1))\frac{d^4p'}{(2\pi)^4} - ig_1{}^2\gamma_\nu\gamma^5\int \mathrm{Tr}\,(\gamma^\nu\gamma^5 iS_F(p'+\tfrac{1}{2}q, m_1)^1 \times\\
&\quad\Gamma_1^{\mu5}(p'+\tfrac{1}{2}q, p'-\tfrac{1}{2}q)iS_F(p'-\tfrac{1}{2}q, m_1))\frac{d^4p'}{(2\pi)^4} - if^2\gamma_\nu\gamma^5\\
&\quad\times\int \mathrm{Tr}\,(\gamma^\nu\gamma^5 iS_F(p'+\tfrac{1}{2}q, m_2)^1\Gamma_2^{\mu5}(p'+\tfrac{1}{2}q, p'-\tfrac{1}{2}q)\\
&\quad\times iS_F(p'-\tfrac{1}{2}q, m_2))\frac{d^4p'}{(2\pi)^4}
\end{aligned}
\tag{3.4a}
$$

$$
\begin{aligned}
&f'\Gamma_2^{\mu5}(p+\tfrac{1}{2}q, p-\tfrac{1}{2}q)\\
&\quad= f\gamma^\mu\gamma^5 + 2ig_2f\gamma^5\int \mathrm{Tr}\,(\gamma^5 iS_F(p'+\tfrac{1}{2}q, m_2)^1\Gamma_2^{\mu5}(p'+\tfrac{1}{2}q, p'-\tfrac{1}{2}q)\\
&\quad\times iS_F(p'-\tfrac{1}{2}q, m_2))\frac{d^4p'}{(2\pi)^4} - ig_2f\gamma_\nu\gamma^5\int \mathrm{Tr}\,(\gamma^\nu\gamma^5\, iS_F(p'+\tfrac{1}{2}q, m_2)\\
&\quad\times{}^1\Gamma_2^{\mu5}(p'+\tfrac{1}{2}q, p'-\tfrac{1}{2}q)iS_F(p'-\tfrac{1}{2}q, m_2))\frac{d^4p'}{(2\pi)^4} - ifg_1\gamma_\nu\gamma^5\int \mathrm{Tr}\,(\gamma^\nu\gamma^5 i\times\\
&\quad S_F(p'+\tfrac{1}{2}q, m_1)^1\Gamma_1^{\mu5}(p'+\tfrac{1}{2}q, p'-\tfrac{1}{2}q)iS_F(p'-\tfrac{1}{2}q, m_1))\frac{d^4p'}{(2\pi)^4}
\end{aligned}
\tag{3.4b}
$$

We make the same ansatz for the vertex functions as in equation (2.31)

$$^1\Gamma_1^{\mu 5}(p+\tfrac{1}{2}q, p-\tfrac{1}{2}q) = \left(\gamma^\mu - \frac{2m_1 q^\mu}{q^2}\right)\gamma^5\, {}^1F_1(q^2) + \left(\gamma^\mu - \frac{\not{q} q^\mu}{q^2}\right)\gamma^5\, {}^1G_1(q^2) \tag{3.5a}$$

$$^1\Gamma_2^{\mu 5}(p+\tfrac{1}{2}q, p-\tfrac{1}{2}q) = \left(\gamma^\mu - \frac{2m_2 q^\mu}{q^2}\right)\gamma^5\, {}^1F_2(q^2) + \left(\gamma^\mu - \frac{\not{q} q^\mu}{q^2}\right)\gamma^5\, {}^1G_2(q^2) \tag{3.5b}$$

On substituting equations (3.5) into equations (3.4),and using equations (2.5), (2.10), (2.23), and (2.27) for particles 1 and 2, we obtain

$$\begin{aligned}&\left(\gamma^\mu - \frac{2m_1 q^\mu}{q^2}\right)\gamma^5\, {}^1F_1(q^2) + \left(\gamma^\mu - \frac{\not{q} q^\mu}{q^2}\right)\gamma^5\, {}^1G_1(q^2)\\ &= \gamma^\mu\gamma^5 - \frac{2m_1 q^\mu}{q^2}\gamma^5\, {}^1F_1(q^2) + \left(\gamma^\mu - \frac{\not{q} q^\mu}{q^2}\right)\gamma^5\left[J_{A_1}(q^2)({}^1F_1(q^2) + {}^1G_1(q^2))\right.\\ &\left.\quad + \frac{f^2}{g_1 g_2} J_{A_2}(q^2)({}^1F_2(q^2) + {}^1G_2(q^2))\right]\end{aligned} \tag{3.6a}$$

$$\begin{aligned}&\left(\gamma^\mu - \frac{2m_2 q^\mu}{q^2}\right)\gamma^5\, {}^1F_2(q^2) + \left(\gamma^\mu - \frac{\not{q} q^\mu}{q^2}\right)\gamma^5\, {}^1G_2(q^2)\\ &= \gamma^\mu\gamma^5 - \frac{2m_2 q^\mu}{q^2}\gamma^5\, {}^1F_2(q^2) + \left(\gamma^\mu - \frac{\not{q} q^\mu}{q^2}\right)\gamma^5\, [J_{A_2}(q^2)({}^1F_2(q^2) + {}^1G_2(q^2))\\ &\quad + J_{A_1}(q^2)({}^1F_1(q^2) + {}^1G_1(q^2))]\end{aligned} \tag{3.6b}$$

The equations (3.6) immediately give:

$$^1F_1(q^2) = 1 \tag{3.7a}$$

$$^1F_2(q^2) = 1 \tag{3.7b}$$

and on substituting equations (3.7) into equations (3.6), a little algebra enables to solve for ${}^1G_1(q^2)$ and ${}^1G_2(q^2)$:

$$^1G_1(q^2) = \frac{J_{A_1}(q^2) + \dfrac{f^2}{g_1 g_2} J_{A_2}(q^2) - \left(1 - \dfrac{f^2}{g_1 g_2}\right) J_{A_1}(q^2) J_{A_2}(q^2)}{1 - J_{A_1}(q^2) - J_{A_2}(q^2) + \left(1 - \dfrac{f^2}{g_1 g_2}\right) J_{A_1}(q^2) J_{A_2}(q^2)} \tag{3.8a}$$

$$^1G_2(q^2) = \frac{J_{A_1}(q^2) + J_{A_2}(q^2) - \left(1 - \dfrac{f^2}{g_1 g_2}\right) J_{A_1}(q^2) J_{A_2}(q^2)}{1 - J_{A_1}(q^2) - J_{A_2}(q^2) + \left(1 - \dfrac{f^2}{g_1 g_2}\right) J_{A_1}(q^2) J_{A_2}(q^2)} \tag{3.8b}$$

Putting together equations (3.5), (3.7), and (3.8), we find that when the external particle lines are on the mass-shell we have:†

$$\begin{aligned}
&\bar{u}_1(p + \tfrac{1}{2}q)\,{}^1\Gamma_1^{\mu 5}(p + \tfrac{1}{2}q, p - \tfrac{1}{2}q) u_1(p - \tfrac{1}{2}q) \\
&\quad = \bar{u}_1(p + \tfrac{1}{2}q)\left(\gamma^\mu - \frac{2m_1 q^\mu}{q^2}\right)\gamma^5 u_1(p - \tfrac{1}{2}q) \\
&\quad \times \left(\frac{1 - \left(1 - \dfrac{f^2}{g_1 g_2}\right) J_{A_2}(q^2)}{1 - J_{A_1}(q^2) - J_{A_2}(q^2) + \left(1 - \dfrac{f^2}{g_1 g_2}\right) J_{A_1}(q^2) J_{A_2}(q^2)}\right)
\end{aligned} \tag{3.9a}$$

$$\begin{aligned}
&\bar{u}_2(p + \tfrac{1}{2}q)\,{}^1\Gamma_2^{\mu 5}(p + \tfrac{1}{2}q, p - \tfrac{1}{2}q) u_2(p - \tfrac{1}{2}q) \\
&\quad = \bar{u}_2(p + \tfrac{1}{2}q)\left(\gamma^\mu - \frac{2m_2 q^\mu}{q^2}\right)\gamma^\mu u_2(p - \tfrac{1}{2}q) \\
&\quad \times \left(\frac{1}{1 - J_{A_1}(q^2) - J_{A_2}(q^2) + \left(1 - \dfrac{f^2}{g_1 g_2}\right) J_{A_1}(q^2) J_{A_2}(q^2)}\right)
\end{aligned} \tag{3.9b}$$

An exactly similar procedure, starting with the pair of coupled integral equations for ${}^2\Gamma_1^{\mu 5}$ and ${}^2\Gamma_2^{\mu 5}$ gives finally:

$$\begin{aligned}
&\bar{u}_1(p + \tfrac{1}{2}q)\,{}^2\Gamma_1^{\mu 5}(p + \tfrac{1}{2}q, p - \tfrac{1}{2}q) u_1(p - \tfrac{1}{2}q) \\
&\quad = \bar{u}_1(p + \tfrac{1}{2}q)\left(\gamma^\mu - \frac{2m_1 q^\mu}{q^2}\right)\gamma^5 u_1(p - \tfrac{1}{2}q) \\
&\quad \times \left(\frac{1}{1 - J_{A_1}(q^2) - J_{A_2}(q^2) + \left(1 - \dfrac{f^2}{g_1 g_2}\right) J_{A_1}(q^2) J_{A_2}(q^2)}\right)
\end{aligned} \tag{3.10a}$$

† The condition $f^2/g_1 g_2 = 1$ corresponds to the special case when the three axial-vector interactions are mediated by a single axial-vector meson in the limit of large meson mass.

$$\bar{u}_2(p+\tfrac{1}{2}q)\,{}^2\Gamma_2^{\mu 5}(p+\tfrac{1}{2}q, p-\tfrac{1}{2}q)u_2(p-\tfrac{1}{2}q)$$

$$= \bar{u}_2(p+\tfrac{1}{2}q)\left(\gamma^\mu - \frac{2m_2 q^\mu}{q^2}\right)\gamma^5 u_2(p-\tfrac{1}{2}q)$$

$$\times \left(\frac{1 - \left(1 - \dfrac{f^2}{g_1 g_2} J_{A_1}(q^2)\right)}{1 - J_{A_1}(q^2) - J_{A_2}(q^2) + \left(1 - \dfrac{f^2}{g_1 g_2}\right) J_{A_1}(q^2) J_{A_2}(q^2)} \right) \tag{3.10b}$$

We therefore see that the four axial-vector vertex functions ${}^1\Gamma_1^{\mu 5}$, ${}^1\Gamma_2^{\mu 5}$, ${}^2\Gamma_1^{\mu 5}$, and ${}^2\Gamma_2^{\mu 5}$ all have the same singularity structure, viz. the pole at $q^2 = 0$ associated with a pseudoscalar meson, and the denominator of the form-factors which will be investigated shortly. This strongly suggests, but does not actually prove, that the poles at $q^2 = 0$ all come from the *same* pseudoscalar meson, i.e. that although there are two spontaneously broken symmetries (3.2), there is only one massless Nambu-Goldstone boson in the theory. This will be shown explicitly in Section 3C.

We now turn our attention to the denominator $D(q^2)$ of the four form-factors:

$$D(q^2) = (1 - J_{A_1}(q^2))(1 - J_{A_2}(q^2)) - \frac{f^2}{g_1 g_2} J_{A_1}(q^2) J_{A_2}(q^2) \tag{3.11}$$

The dispersive form for $J_{A_a}(q^2)$ is given by equation (2.25a) and the dispersive form for $(1 - J_{A_a}(q^2))$ is given by equation (2.25b). On substituting into equation (3.11) we obtain:

$$D(q^2) = \frac{g_1 g_2}{144\pi^4} \left(\int_{4m_1^2}^{\Lambda_1^2} d\kappa_1^2 \frac{(1 - 4m_1^2/\kappa_1^2)^{1/2}(4(\kappa_1^2 - m_1^2) - 3q^2)}{\kappa_1^2 - q^2} \right)$$

$$\times \left(\int_{4m_2^2}^{\Lambda_2^2} d\kappa_2^2 \frac{(1 - 4m_2^2/\kappa_2^2)^{1/2}(4(\kappa_2^2 - m_2^2) - 3q^2)}{\kappa_2^2 - q^2} \right)$$

$$- \frac{g_1 g_2}{144\pi^4} \frac{f^2}{g_1 g_2} \left(\int_{4m_1^2}^{\Lambda_1^2} d\kappa_1^2 \frac{(1 - 4m_1^2/\kappa_1^2)^{1/2}(4m_1^2 - \kappa_1^2)}{\kappa_1^2 - q^2} \right)$$

$$\times \left(\int_{4m_2^2}^{\Lambda_2^2} d\kappa_2^2 \frac{(1 - 4m_2^2/\kappa_2^2)^{1/2}(4m_2^2 - \kappa_2^2)}{\kappa_2^2 - q^2} \right) \tag{3.12}$$

It is clear that $D(q^2)$ has branch cuts from $4m_1^2$ to Λ_1^2 and from $4m_2^2$ to Λ_2^2. To see if it has any zeros as a function of q^2, we combine the products of integrals in equation (3.12) into a double integral:

$$D(q^2) = \frac{g_1 g_2}{144\pi^4} \int_{4m_1^2}^{\Lambda_1^2} d\kappa_1^2 \int_{4m_2^2}^{\Lambda_2^2} d\kappa_2^2 \frac{(1 - 4m_1^2/\kappa_1^2)^{1/2}(1 - 4m_2^2/\kappa_2^2)^{1/2}}{(\kappa_1^2 - q^2)(\kappa_2^2 - q^2)}$$

$$\times \left[\begin{array}{c} (4(\kappa_1^2 - m_1^2) - 3q^2)(4(\kappa_2^2 - m_2^2) - 3q^2) \\ - \dfrac{f^2}{g_1 g_2}(4m_1^2 - \kappa_1^2)(4m_2^2 - \kappa_2^2) \end{array} \right] \tag{3.13}$$

so that the integrand will be zero when the expressions within square brackets is zero:

$$(4(\kappa_1^2 - m_1^2) - 3q^2)(4(\kappa_2^2 - m_2^2) - 3q^2) = \frac{f^2}{g_1 g_2}(4m_1^2 - \kappa_1^2)(4m_2^2 - \kappa_2^2) \tag{3.14}$$

It is, of course, trivial to solve this equation for q^2 explicitly. It is, however, more illuminating to plot the left-hand side and right-hand side on a graph, assuming without loss of generality that $m_1^2 < m_2^2$ (see Fig. 8). It is seen that as f^2/g_1g_2 increases from zero to infinity, for fixed κ_1^2 and κ_2^2, the lower zero moves down the q^2-axis from $4(\kappa_1^2 - m_1^2)/3$ to $-\infty$. So the form factors, whose denominator $D(q^2)$ is given by equation (3.13), will have *a pole corresponding to an axial-vector particle* which, for small values of f^2/g_1g_2 is an unstable particle, for medium values of f^2/g_1g_2 is a stable particle, and for large values of f^2/g_1g_2 is a tachyon.† The actual mass of the particle for a

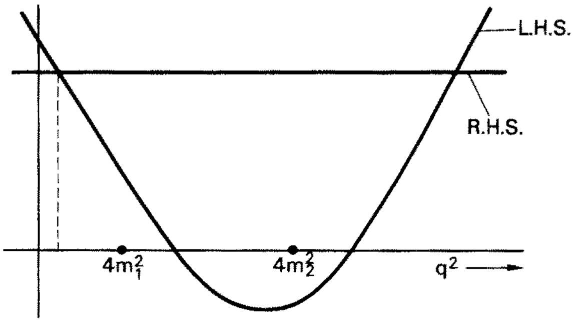

Figure 8.—Plot of equation (3.14) showing a solution in the range $0 < q^2 < 4m_1^2$.

† The presence of a tachyon in the limit of large coupling is, of course, a well-known feature of mixing theories. It occurs for example in the theory (Deo, 1961) described by

$$\mathscr{L} = \tfrac{1}{2}(\partial\varphi_1)^2 + \tfrac{1}{2}(\partial\varphi_2)^2 - \frac{\mu_1^2}{2}\varphi_1^2 - \frac{\mu_2^2}{2}\varphi_2^2 + g\varphi_1\varphi_2$$

when $g^2 > \mu_1^2\mu_2^2$. For $g^2 \gg \mu_1^2\mu_2^2$, this theory becomes equivalent to that described by equation (1.1).

given value of f^2/g_1g_2 is, of course, dependent upon the cut-off masses Λ_1^2 and Λ_2^2. The higher zero corresponds to an unstable particle, whose mass increases as f^2/g_1g_2 increases. When $f^2/g_1g_2 = 0$ the two are at $q^2 = 4(\kappa_a^2 - m_a^2)/3$ as we would expect from Section 2B and 2C (see equation (2.37) and the discussion following equation (2.25)), and ${}^1\Gamma_1^{\mu5}$ and ${}^2\Gamma_2^{\mu5}$ have form factors $1/(1 - J_{A_1}(q^2))$ and $1/(1 - J_{A_2}(q^2))$ respectively, on the fermion mass-shell. We have attempted to evaluate the integrals in equation (3.12) using the substitution $\kappa_a^2 = (y_a + m_a^2)^2/y_a$ but this procedure yields a rather complicated expression for $D(q^2)$ which is not very informative.

As mentioned in Section 2B, when we use Nambu and Jona-Lasinio's dispersion integral for $J_{A_a}(q^2)$ (i.e. with $J_{V_a}(0) = 0$, the expression which we obtain for $D(q^2)$ does not appear to make physical sense when we consider the behaviour of the zeros of $D(q^2)$ for different values of f^2/g_1g_2. This is discussed in Appendix B.

C. Coupling of Strings of Pseudoscalar Bubbles

We saw in Sections 2B and 2C that, in the case of a single Nambu-Jona-Lasinio field, the Nambu-Goldstone boson associated with spontaneous break-

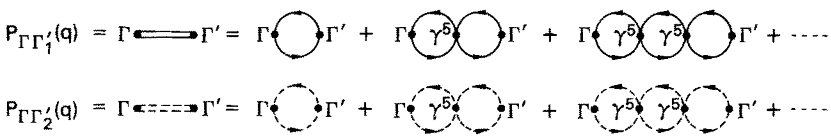

Figure 9.—The functions $P_{\Gamma\Gamma_1'}(q)$ and $P_{\Gamma\Gamma_2'}(q)$ for $\Gamma\Gamma' = AA, AP$, or PA. For $\Gamma\Gamma' = PP$, there is an extra no-loop term in the sums.

down of chiral symmetry is the massless pseudoscalar meson generated by the sum of all possible strings of pseudoscalar bubbles $J_{PP}(q)$. In this section we shall study what happens when the strings of pseudoscalar bubbles associated with particles 1 and 2 are allowed to mix via the term $f(\bar{\psi}_1\gamma^\mu\gamma^5\psi_1) \times (\bar{\psi}_2\gamma_\mu\gamma^5\psi_2)$ in the Lagrangian (3.1), in the same spirit as we discussed the mixing Lagrangians (1.1) and (1.3) in the Introduction.

We first of all define $P_{\Gamma\Gamma_a'}(q)$ (where $a = 1, 2$) as the sum of all possible strings of pseudoscalar bubbles of particle a with Γ and Γ' at ends (see Fig. 9). We find for $P^{\mu\nu}_{AA_a}(q)$ that:

$$P^{\mu\nu}_{AA_a}(q) = \tilde{J}^{\mu\nu}_{AA_a}(q) - 2ig_a \frac{\tilde{J}^{\mu}_{AP_a}(q)\tilde{J}^{\nu}_{PA_a}(q)}{1 - J_{PP_a}(q)} \tag{3.15}$$

which, on using equations (2.11), (2.23), (2.27), and (2.28), gives

$$P^{\mu\nu}_{AA_a}(q) = -i\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right)\frac{J_{A_a}(q^2)}{g_a} \tag{3.16}$$

which has a pole at $q^2 = 0$.† (Note, incidentally, that $P^{\mu\nu}_{AA_2}(q)$ which consists of a stream of *particle 2* pseudoscalar bubbles, gives a contribution to $g_1\bar{u}_1(p + \frac{1}{2}q)^1\Gamma_1^{\mu 5}(p + \frac{1}{2}q, p - \frac{1}{2}q)u_1(p - \frac{1}{2}q)$ of the form $(\gamma^\mu - 2m_1 q^\mu/q^2) \times \gamma^5(f^2/g_2)J_{A_2}(q^2)$.) Similarly, we find for $P^\mu_{AP_a}(q)$

$$P^\mu_{AP_a}(q) = \frac{\tilde{J}^\mu_{AP_a}(q)}{1 - J_{PP_a}(q)} \tag{3.17}$$

which, on using equations (2.11) and (2.28), gives

$$P^\mu_{AP_a}(q) = \frac{im_a q^\mu}{g_a q^2} \tag{3.18}$$

Furthmore, since $\tilde{J}^\mu_{PA_a}(q) = -\tilde{J}^\mu_{AP_a}(q)$, we find

$$P^\mu_{PA_a}(q) = -\frac{im_a q^\mu}{g_a q^2} \tag{3.19}$$

Both $P^\mu_{AP_a}(q)$ and $P^\mu_{PA_a}(q)$ have a pole at $q^2 = 0$. The sum of pseudoscalar bubbles $P_{PP_a}(q)$ is equal to $1/(1 - J_{PP_a}(q))$ which, by equation (2.11), gives

$$P_{PP_a}(q) = -\frac{i}{4g_a q^2 I_a(q^2)} \tag{3.20}$$

and it too has a pole at $q^2 = 0$.

We now consider what happens when we connect these strings of bubbles with the coupling $f(\bar{\psi}_1\gamma^\mu\gamma^5\psi_1)(\bar{\psi}_2\gamma_\mu\gamma^5\psi_2)$ to form the functions $S_{\Gamma\Gamma'_{ab}}(q)$ shown in Fig. 10. It is obvious, since $(g^{\mu\nu} - q^\mu q^\nu/q^2)$ is indempotent, $(g^{\mu\rho} - q^\mu q^\rho/q^2)(g_\rho{}^\nu - q_\rho q^\nu/q^2) = (g^{\mu\nu} - q^\mu q^\nu/q^2)$, that the functions $S^{\mu\nu}_{AA_{ab}}(q)$ have only a single pole at $q^2 = 0$. In fact, we find

$$S^{\mu\nu}_{AA_{11}}(q) = -i\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right)\left(\frac{J_{A_1}(q^2)/g_1}{1 - \dfrac{f^2}{g_1 g_2}J_{A_1}(q^2)J_{A_2}(q^2)}\right) \tag{3.21a}$$

$$S^{\mu\nu}_{AA_{22}}(q) = -i\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right)\left(\frac{J_{A_2}(q^2)/g_2}{1 - \dfrac{f^2}{g_1 g_2}J_{A_1}(q^2)J_{A_2}(q^2)}\right) \tag{3.21b}$$

$$S^{\mu\nu}_{AA_{12}}(q) = S^{\mu\nu}_{AA_{21}}(q) = -i\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right)\left(\frac{fJ_{A_1}(q^2)J_{A_2}(q^2)/g_1 g_2}{1 - \dfrac{f^2}{g_1 g_2}J_{A_1}(q^2)J_{A_2}(q^2)}\right) \tag{3.21c}$$

† Note that, from equation (2.25a), $J_A(0) < 0$ since $g > 0$, (i.e. it is non-zero and finite).

and it should be clear from Fig. 10 that the pole at $q^2 = 0$ in equations (3.2) comes from the same massless boson in all four cases. The denominator $(1 - (f^2/g_1g_2)J_{A_1}(q^2)J_{A_2}(q^2))$ of the 'form-factor' is different from that in equations (3.9) and (3.10). This is because in this section we have ignored the axial-vector coupling of loops of the same particle. If, in Section 3B, we had approximated the kernels K_{11} and K_{22} by the pseudoscalar interaction only, then this denominator would have occurred in the form factors of the axial-vector vertices. It still gives two poles, of the same general kind as the denominator of Section 3B (see Fig. 8).

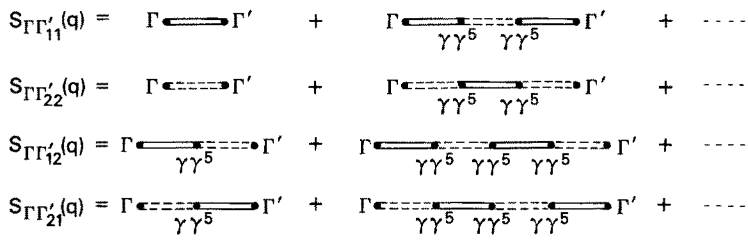

Figure 10.—The functions $S_{\Gamma\Gamma'_{11}}(q)$, $S_{\Gamma\Gamma'_{22}}(q)$, $S_{\Gamma\Gamma'_{12}}(q)$, and $S_{\Gamma\Gamma'_{21}}(q)$.

On evaluating $S^{\mu}_{AP_{ab}}(q)$ we find, since $(q^{\mu}/q^2)(g_{\mu\nu} - q_{\mu}q_{\nu}/q^2) = 0$, that only the first terms in the series for $S^{\mu}_{AP_{11}}(q)$ and $S^{\mu}_{AP_{22}}(q)$ contribute, and that $S^{\mu}_{AP_{12}}(q)$ is zero:

$$S^{\mu}_{AP_{11}}(q) = P^{\mu}_{AP_{11}}(q) = \frac{im_1q^{\mu}}{g_1q^2} \tag{3.22a}$$

$$S^{\mu}_{AP_{22}}(q) = P^{\mu}_{AP_{22}}(q) = \frac{im_2q^{\mu}}{g_2q^2} \tag{3.22b}$$

$$S^{\mu}_{AP_{12}}(q) = 0 = S^{\mu}_{AP_{21}}(q) \tag{3.22c}$$

with similar expressions for $S^{\mu}_{PA_{ab}}(q)$. The fact that the poles at $q^2 = 0$ in equations (3.22a) and (3.22b) come only from streams of pseudoscalar bubbles of particles 1 and 2 respectively does not invalidate our claim that there is only one massless boson in the theory. It is just that in the particular Green functions $S^{\mu}_{AP_{11}}(q)$ and $S^{\mu}_{AP_{22}}(q)$ there is no explicit mixing. But in any process to which they contribute, e.g. meson exchange in the process $21 \to 21$, there will also be a contribution from $S^{\mu\nu}_{AA_{12}}(q)$ in which explicit mixing occurs.

We find, on evaluating $S_{PP_{ab}}(q)$, that all terms except the first are zero, and so

$$S_{PP_{11}}(q) = P_{PP_1}(q) = -\frac{i}{4g_1q^2I_1(q^2)} \tag{3.23a}$$

$$S_{PP_{22}}(q) = P_{PP_2}(q) = -\frac{i}{4g_2q^2I_2(q^2)} \tag{3.23b}$$

$$S_{PP_{12}}(q) = S_{PP_{21}}(q) = \frac{ifm_1m_2}{g_1g_2q^2} \tag{3.23c}$$

All terms have a single pole at $q^2 = 0$, but explicit mixing occurs only in $S_{PP_{12}}(q)$ and $S_{PP_{21}}(q)$.

We have thus shown that in all the Green functions $S_{\Gamma\Gamma'_{ab}}(q)$ which are non-zero there is a single pole at $q^2 = 0$ coming from one massless pseudoscalar meson. Note incidentally, that Umezawa (1965a, 1965b) has shown that the axial-vector current $\bar{\psi}_a\gamma^\mu\gamma^5\psi_a$ can be written in terms of the physical fields $\psi_a^{(m_a)}$ (of the massive fermion) and φ_a (of the massless pseudoscalar meson) as $(g^{\mu\nu} - \partial^\mu\partial^\nu/\Box)\bar{\psi}_a^{(m_a)}\gamma_\nu\gamma^5\psi_a^{(m_a)} + c_a\partial^\mu\varphi_a$ (where c_a is a certain constant). So, as far as interactions between the two pseudoscalar mesons are concerned, the Lagrangian (3.1) gives rise to an effective Lagrangian of the same form as equation (1.3).

The single massless pseudoscalar particle can then presumably be removed from the theory by coupling a massless axial-vector gauge field to either or both of the fermions. The gauge field then acquires a mass by the Higgs mechanism as described by Freundlich & Lurié (1970) and Aurilia *et al.* (1972) (see also Jackiw & Johnson (1973) and Cornwall & Norton (1973)). In our theory, if the gauge field is coupled to say particle 1, then it will acquire a mass by having $S^{\mu\nu}_{AA_{11}}(q)$ as its 'vacuum polarisation tensor'. We have not, however, studied this question yet.

4. *Conclusion*

We have demonstrated a new way for reducing from two to one the number of massless Nambu–Goldstone bosons which occur when two symmetries are spontaneously broken. The method is to allow mixing between the Goldstone bosons. In the model considered, the two unstable axial-vector mesons, which are present in the theory even when there is mixing ($f = 0$), change their form when we allow mixing ($f \neq 0$), the one with larger mass becoming more massive, and the one with smaller mass becoming less massive. We note that if we add another Nambu-Jona-Lasinio field to the Lagrangian density equation (3.1) with a mixing term of the form say $h(\bar{\psi}_2\gamma^\mu\gamma^5\psi_2) \times (\bar{\psi}_3\gamma_\mu\gamma^5\psi_3)$ then the three chiral symmetries of the theory are realised by only one massless pseudoscalar boson as can be seen by considering the diagram of Fig. 11 which is equal to

$$P^{\mu\rho}_{AA_1}(q)P_{AA_2\rho\sigma}(q)P^{\sigma\nu}_{AA_3}(q) = -i\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right)\frac{fh}{g_1g_2g_3}J_{A_1}(q^2)J_{A_2}(q^2)J_{A_3}(q^2) \tag{4.1}$$

On summing all possible string diagrams with an axial-vector particle 1 vertex at one end, and an axial-vector particle 3 vertex at the other, we would obtain

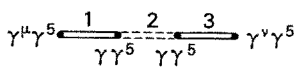

Figure 11.—The coupling of three strings of pseudoscalar bubbles in the theory with three Nambu–Jona-Lasinio fields.

a Green function with a single pseudoscalar pole at $q^2 = 0$ and three poles corresponding to axial-vector mesons. (Exactly similar considerations apply to the model of equation (1.3) when we add an extra field φ_3 with mixing term $h(\partial\varphi_2).(\partial\varphi_3)\gamma$. Thus we apparently can have any number of Nambu-Goldstone symmetries realised by only one massless spin-zero boson, if we allow mixing between them.

As for physical applications,† we cannot think of any off hand, but we may remark that the ideas expressed in this paper appear to be a useful tool to keep in the workshop when model-building. Another point we would like to emphasise is the natural appearnace to tachyons in mixing theories with large coupling constants. Some people are disturbed by the appearance of such fields in the 'undisplaced' form of the models of Goldstone (1961) and Higgs (1966), but it seems to us very significant that a Lagrangian of the type mentioned in footnote† (p. 123), which describes such an everyday occurrence as $\omega - \phi$ mixing, in the limit of large $|g|$ gives a tachyon.

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I thank Dr. D. J. Wallace for an interesting conversation during the preliminary stages of this work.

Appendix A

In this Appendix we shall list a few of the expressions used in evaluating the bubble graphs and for converting them into dispersion integrals.

1. *An Expression for*

$$\int \frac{p^2}{((p-\frac{1}{2}q)^2 - m^2)((p+\frac{1}{2}q)^2 - m^2)} \frac{d^4p}{(2\pi)^4}$$

The expression

$$\int \frac{d^4p}{p^2 - m^2} - \int \frac{d^4p \qquad p^2}{((p-\frac{1}{2}q)^2 - m^2)((p+\frac{1}{2}q)^2 - m^2)}$$

$$= \int \frac{d^4p \quad ((p-q)^2 - m^2)}{((p-q)^2 - m^2)(p^2 - m^2)} - \int \frac{d^4p \qquad (p-\frac{1}{2}q)^2}{((p-q)^2 - m^2)(p^2 - m^2)} \qquad \text{(A.1)}$$

† It may be of interest to note that the original inspiration for this paper was the author's naive belief that a crystal with two different atoms per unit cell is an example of a system with two spontaneously broken symmetries giving rise to one massless boson (the acoustical phonon) and one massive boson (the optical phonon). However, this system is described by a Lagrangian density whose relativistic analogue is $\mathscr{L} = \frac{1}{2}(\partial\varphi_1)^2 + \frac{1}{2}(\partial\varphi_2)^2 + g(\varphi_1 - \varphi_2)^2$ and is invariant only under *one* displacement transformation:

$$\varphi_1(x) \to \varphi_1(x) + \lambda, \varphi_2(x) \to \varphi_2(x) + \lambda.$$

where we have displaced the integration variable $p \to p - \frac{1}{2}q$ in the second integral. On adding the numerators, this gives

$$\int \frac{d^4p}{p^2 - m^2} - \int \frac{d^4p}{((p - \frac{1}{2}q)^2 - m^2)((p + \frac{1}{2}q)^2 - m^2)} \frac{p^2}{}$$

$$= \int \frac{d^4p(-p\,.\,q + \frac{3}{4}q^2 - m^2)}{((p - q)^2 - m^2)(p^2 - m^2)} \quad \text{(A.2)}$$

and on displacing the integration variable back again $p \to p + \frac{1}{2}q$, and using the fact that $\int d^4pf(p) = 0$ for $f(-p) = -f(p)$, we obtain

$$\int \frac{d^4p}{p^2 - m^2} - \int \frac{d^4p \; p^2}{((p - \frac{1}{2}q)^2 - m^2)((p + \frac{1}{2}q)^2 - m^2)}$$

$$= \left(\frac{q^2}{4} - m^2\right) \int \frac{d^4p}{((p - \frac{1}{2}q)^2 - m^2)((p + \frac{1}{2}q)^2 - m^2)} \quad \text{(A.3)}$$

which, on rearrangement, gives

$$\int \frac{p^2}{((p - \frac{1}{2}q)^2 - m^2)((p + \frac{1}{2}q)^2 - m^2)} \frac{d^4p}{(2\pi)^4}$$

$$= \frac{1}{(2\pi)^4} \int \frac{d^4p}{p^2 - m^2} - \left(\frac{q^2}{4} - m^2\right)$$

$$\times \int \frac{1}{((p - \frac{1}{2}q)^2 - m^2)((p + \frac{1}{2}q)^2 - m^2)} \frac{d^4p}{(2\pi)^4}$$

which we rewrite as:

$$\int \frac{p^2}{((p - \frac{1}{2}q)^2 - m^2)((p + \frac{1}{2}q)^2 - m^2)} \frac{d^4p}{(2\pi)^4}$$

$$\equiv \frac{1}{(2\pi)^4} \int \frac{d^4p}{p^2 - m^2} - \left(\frac{q^2}{4} - m^2\right) I(q^2) \quad \text{(A.4)}$$

2. *An Expression for* $\int d^4pf(p\,.\,q)p^\mu p^\nu$ *in Terms of One Unknown Function*

It is of course clear that $\int d^4pf(p\,.\,q)p^\mu p^\nu$ is zero for $f(-p\,.\,q) = -f(p\,.\,q)$. For $f(-p\,.\,q) = +f(p\,.\,q)$ the integral must be of the form

$$\int d^4pf(p\,.\,q)p^\mu p^\nu = A(q^2)q^\mu q^\nu + B(q^2)g^{\mu\nu} \tag{A.5}$$

where $A(q^2)$ and $B(q^2)$ are unknown functions. On contracting indices, we find

$$\int d^4pf(p\,.\,q)p^2 = A(q^2)q^2 + 4B(q^2) \tag{A.6}$$

which therefore gives:

$$\int d^4pf(p\,.\,q)p^\mu p^\nu = \tfrac{1}{4}g^{\mu\nu} \int d^4pf(p\,.\,q)p^2 + A(q^2)(q^\mu q^\nu - \tfrac{1}{4}q^2 g^{\mu\nu}) \tag{A.7}$$

which is the required expression.

3. *Dispersion Integrals*

A Feynman integral can be expressed as a dispersion integral by using the Cutkosky rules (Cutkosky, 1960) to obtain the discontinuity across the real axis. These are the same as the Feynman rules except that instead of a propagator denominator $1/(p^2 - m^2 + i\epsilon)$ we have a factor $2\pi i\delta(p^2 - m^2)$. Hence, to express $I(q^2)$ (which physically is the Feynman integral for a loop with spin zero particles of mass m as internal lines) as a dispersion integral, we first find disc $I(q^2)$:

$$\text{disc}\, I(q^2) = (2\pi i)^2 \int \frac{d^4p}{(2\pi)^4}\,\delta((p - \tfrac{1}{2}q)^2 - m^2)\delta((p + \tfrac{1}{2}q)^2 - m^2) \tag{A.8}$$

which is most easily evaluated in the frame where $q^\mu = ((q^2)^{1/2}, 0)$ to give:

$$\text{disc}\, I(q^2) = \frac{(2\pi i)^2}{(2\pi)^4}\,\frac{\pi}{2}\,(1 - 4m^2/q^2)^{1/2} \tag{A.9}$$

and, on writing a dispersion relation for $I(q^2)$ with this discontinuity, we obtain:

$$I(q^2) = \frac{i}{16\pi^2} \int_{4m^2}^{\Lambda^2} \frac{(1 - 4m^2/\kappa^2)^{1/2}}{\kappa^2 - q^2}\,d\kappa^2 \tag{A.10}$$

where we cut off the integral at $\kappa^2 = \Lambda^2$ since it is logarithmically divergent.†

† The cut-off Λ^2 is not equal to the cut-off Λ^2 mentioned in Section 2A.

To find $\int d^4p(p^2-m^2)^{-1}$ as a dispersion integral, we note from equation (2.10) that it is just equal to $-4\pi^4\tilde{J}_{PP}(0)$. The discontinuity of $J_{PP}(q^2)$ is given by:

$$\operatorname{disc}\tilde{J}_{PP}(q^2)=(2\pi i)^2\int\frac{d^4p}{(2\pi)^4}(-4p^2+q^2+4m^2)\delta((p-\tfrac{1}{2}q)^2-m^2)$$
$$\times\,\delta((p+\tfrac{1}{2}q)^2-m^2)\qquad\text{(A.11)}$$

which, evaluation in the frame where $q^\mu=((q^2)^{1/2},0)$, gives

$$\operatorname{disc}\tilde{J}_{PP}(q^2)=\frac{(2\pi i)^2}{(2\pi)^4}\pi q^2(1-4m^2/q^2)^{1/2}\qquad\text{(A.12)}$$

so that

$$\tilde{J}_{PP}(q^2)=\frac{i}{8\pi^2}\int_{4m^2}^{\Lambda^2}\frac{\kappa^2(1-4m^2/\kappa^2)^{1/2}}{\kappa^2-q^2}\,d\kappa^2\qquad\text{(A.13)}$$

We therefore find

$$\int\frac{d^4p}{p^2-m^2}=\frac{\pi^2}{2i}\int_{4m^2}^{\Lambda^2}(1-4m^2/\kappa^2)^{1/2}\,d\kappa^2\qquad\text{(A.14)}$$

so that in terms of the dispersion integral the self-consistent mass equation (2.5) reads

$$1=\frac{g}{4\pi^2}\int_{4m^2}^{\Lambda^2}(1-4m^2/\kappa^2)^{1/2}\,d\kappa^2\qquad\text{(A.15)}$$

Equations (A.10) and (A.15) are the only dispersion integrals needed in this paper. It is, of course, also possible to calculate the discontinuity of each bubble $\tilde{J}_{\Gamma\Gamma'}(q)$ directly, as was done by Nambu and Jona-Lasinio.

Appendix B

In this Appendix, we shall exhibit the poles of the form factors in equations (3.9) and (3.10), i.e. the zero of the function $D(q^2)$ of equation (3.11), using the Nambu-Jona-Lasinio (substracted) form for $J_{A_a}(q^2)$. That is, with the notation of Section 2B, we have

$$J_A^{(\mathrm{sub})}(q^2)=-((J_V(q^2)-J_V(0))+J_A'(q^2))=J_A(q^2)+J_V(0)\qquad\text{(B.1)}$$

On using equations (2.20) and (2.25) we obtain

$$J_A^{(\mathrm{sub})}(q^2)=\frac{g}{12\pi^2}\int_{4m^2}^{\Lambda^2}\frac{(1-4m^2/\kappa^2)^{1/2}}{\kappa^2-q^2}\left(6m^2-q^2\left(1+\frac{2m^2}{\kappa^2}\right)\right)d\kappa^2\qquad\text{(B.2a)}$$

$$1 - J_A^{(\mathrm{sub})}(q^2) = \frac{g}{12\pi^2} \int_{4m^2}^{\Lambda^2} \frac{(1 - 4m^2/\kappa^2)^{1/2}}{\kappa^2 - q^2}$$

$$\times \left(3(\kappa^2 - 2m^2) - 2q^2\left(1 - \frac{m^2}{\kappa^2}\right)\right) d\kappa^2 \qquad \text{(B.2b)}$$

So substituting equations (B.2) into equation (3.11), and looking at the integrand of the double integral, we see that it will be zero when

$$\left[3(\kappa_1^2 - 2m_1^2) - 2q^2\left(1 - \frac{m_1^2}{\kappa_1^2}\right)\right]\left[3(\kappa_2^2 - 2m_2^2) - 2q^2\left(1 - \frac{m_2^2}{\kappa_2^2}\right)\right]$$

$$= \frac{f^2}{g_1 g_2}\left[6m_1^2 - q^2\left(1 + \frac{2m_1^2}{\kappa_1^2}\right)\right]\left[6m_2^2 - q^2\left(1 + \frac{2m_2^2}{\kappa_2^2}\right)\right] \qquad \text{(B.3)}$$

The parabolas which constitute the right-hand side and left-hand side of this equation are plotted in Fig. 12 assuming without loss of generality that $m_1^2 < m_2^2$. It is seen that, as f^2/g_1g_2 increases from 0 to ∞, one pole slowly moves from $q^2 = 3(\kappa_1^2 - 2m_1^2)/2(1 - m_1^2/\kappa_1^2)$ to $q^2 = 6m_2^2/(1 + 2m_2^2/\kappa_2^2)$, whilst the other rapidly moves out from $q^2 = 3(\kappa_2^2 - 2m_2^2)/2(1 - m_2^2/\kappa_2^2)$ to ∞, and then moves in again from $-\infty$ to $q^2 = 6m_1^2/(1 + 2m_1^2/\kappa_1^2)$. This seems to us to be a somewhat unphysical behaviour and justifies our assertion, made in Section 2B, that one should not put $J_V(0) = 0$ in this theory.

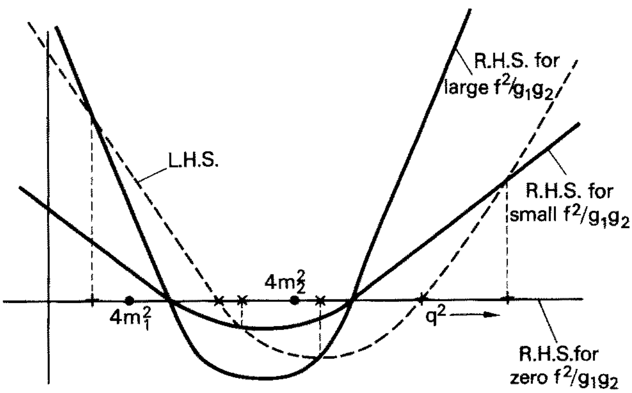

Figure 12.—Plot of equation (B.3), showing the two solutions (denoted by crosses) for $f^2/g_1g_2 = 0$, f^2/g_1g_2 = small, f^2/g_1g_2 = large.

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